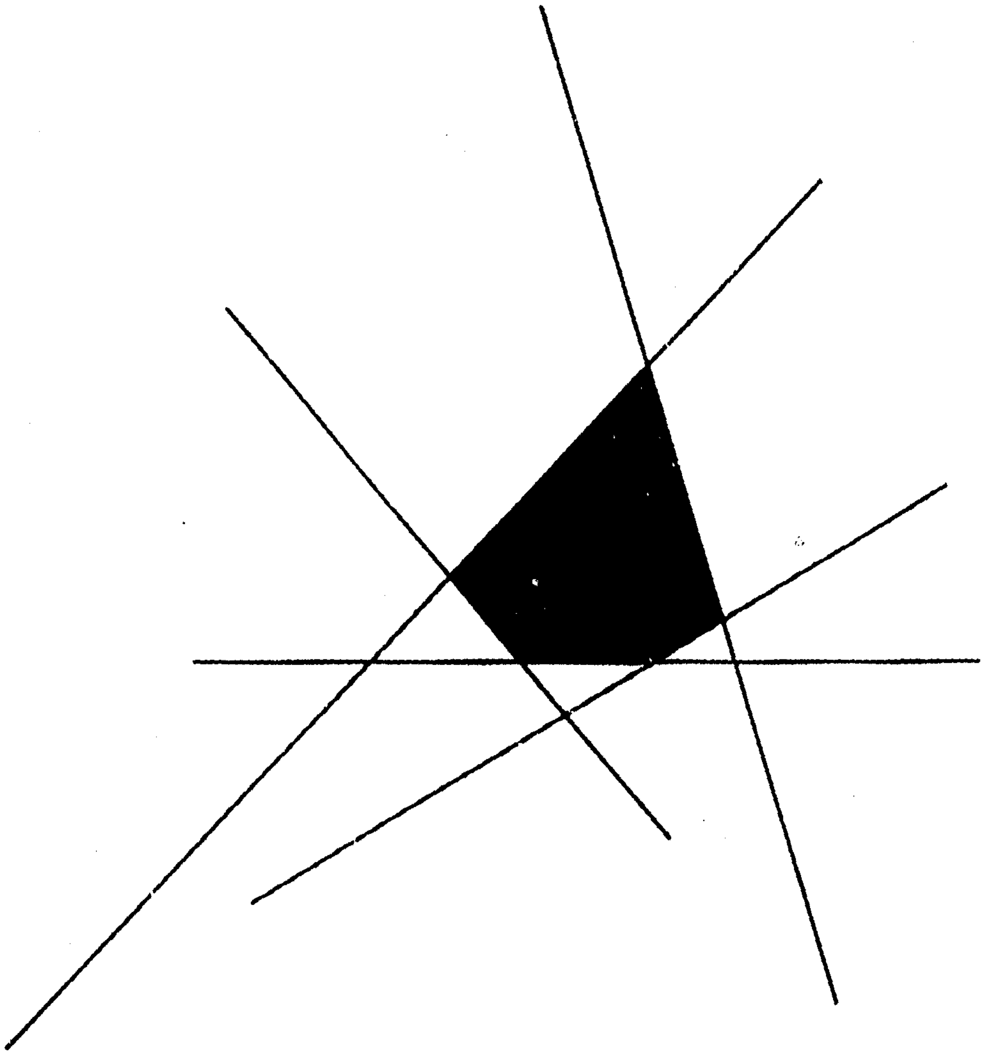
ORC 72-8
APRIL 1972

# AVAILABILITY THEORY FOR MULTICOMPONENT SYSTEMS

by

Richard E. Barlow and Frank Proschan





**OPERATIONS RESEARCH CENTER**



**COLLEGE OF ENGINEERING**
**UNIVERSITY OF CALIFORNIA · BERKELEY**

AVAILABILITY THEORY FOR MULTICOMPONENT SYSTEMS

by

Richard E. Barlow[†]
Department of Industrial Engineering
and Operations Research
University of California, Berkeley

and

Frank Proschan[††]
Department of Statistics
The Florida State University
Tallahassee, Florida

APRIL 1972 ORC 72-8

[†]This research has been partially supported by the Office of Naval Research under Contract N00014-69-A-0200-1036 and the National Science Foundation under Grant GP-29123 with the University of California. Reproduction in whole or in part is permitted for any purpose of the United States Government.

[††]Research sponsored by the Air Force Office of Scientific Research, AFSC, USAF, under Grant AFOSR-71-2058 with Florida State University.

Unclassified

Security Classification

## DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

| | |
|---|---|
| 1. ORIGINATING ACTIVITY (Corporate author)<br>University of California, Berkeley | 2a. REPORT SECURITY CLASSIFICATION<br>Unclassified<br>2b. GROUP |

3. REPORT TITLE

AVAILABILITY THEORY FOR MULTICOMPONENT SYSTEMS

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Research Report

5. AUTHOR(S) (First name, middle initial, last name)

Richard E. Barlow and Frank Proschan

| 6. REPORT DATE | 7a. TOTAL NO. OF PAGES | 7b. NO. OF REFS |
|---|---|---|
| April 1972 | 28 | 12 |

| | |
|---|---|
| 8a. CONTRACT OR GRANT NO<br>N00014-69-A-0200-1036<br>b. PROJECT NO<br>NR 042 238<br>c.<br>Research Project No.: WW 041<br>d. | 9a. ORIGINATOR'S REPORT NUMBER(S)<br>ORC 72-8<br>9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) |

10. DISTRIBUTION STATEMENT

This document has been approved for public release and sale; its distribution is unlimited.

| | |
|---|---|
| 11. SUPPLEMENTARY NOTES Also supported by the Nat'l. Sci. Found., Grant GP-29123, and the Air Force Office of Scientific Research, AFSC, USAF, Grant AFOSR-71-2058. | 12. SPONSORING MILITARY ACTIVITY<br>Statistics & Probability<br>Office of Naval Research<br>Department of the Navy<br>Arlington, Virginia 22217 |

13. ABSTRACT

SEE ABSTRACT.



DD FORM 1 NOV 65 1473 (PAGE 1)

Unclassified

Security Classification

Unclassified

Security Classification

| 14 KEY WORDS | LINK A | | LINK B | | LINK C | |
|---|---|---|---|---|---|---|
| | ROLE | WT | ROLE | WT | ROLE | WT |
| Availability | | | | | | |
| Series System | | | | | | |
| System Up Time | | | | | | |
| Down Time | | | | | | |
| Renewal Processes | | | | | | |



DD FORM 1 NOV 65 1473 (BACK)

S/N 0101-807-6821

Unclassified

Security Classification

A-31409

## ABSTRACT

A series system for which only failed components are replaced (or repaired) is considered. Nonfailed components are in a "state-of-suspended animation" while the system is down. The limiting average system up time is computed for arbitrary failure and repair distributions. The limiting distribution of system up time, the number of failures of component i (i = 1,2, ..., k) and the down time of component i (i = 1,2, ..., k) are calculated. The asymptotic distribution of the cost of repair is also derived.

# AVAILABILITY THEORY FOR MULTICOMPONENT SYSTEMS

by

Richard E. Barlow and Frank Proschan

## 0. INTRODUCTION AND SUMMARY

In this paper we consider one of the most basic stochastic models in reliability theory--namely the on-off process generated by failures and repairs of components in a series system. A series system of $k$ components operates if and only if each of the $k$ components operates. No component operates while the system is down. Furthermore, only failed components are repaired and/or replaced; repair or replacement takes a random time. Repaired components are assumed to function like new components.

There is a large literature dealing with availability, the probability that the system is functioning. However most papers assume special repair or failure distributions (or both). An example of this type is the paper by Gaver (1963) and the paper by Obretenov, Dimitrov and Uzunov (1969). Many papers, including those just named, are concerned with parallel systems with independently operating components - that is nonfailed components are not usually in a state of "suspended animation" during repair of a failed component, as in our model. The reader may consult the IEEE Transactions on Reliability as well as the *Colloquium on Reliability Theory* held at Tihany, Hungary, 16-19 September, 1969. A model more general than ours is treated by M. C. Botez (1969). However there is no overlap with our results.

Let $X_{ir}$ be the duration of the rth functioning period of component $i$ with distribution $F_i$ and mean $\mu_i$ , $i = 1,2, \ldots, k$ (i.e., time to failure of rth replacement for component $i$ excluding system down times). Let $D_{ir}$ be the duration of the repair time (or down time) for component $i$ with distribution $G_i$ and mean $\nu_i$ , $i = 1,2, \ldots, k$ . We assume that for $i = 1, \ldots, k$ ,

$\{X_{ir}\}_{r=1,2,...}$ and $\{D_{ir}\}_{r=1,2,...}$ are mutually independent renewal processes.

A typical failure-repair history for such a series system might look as follows:

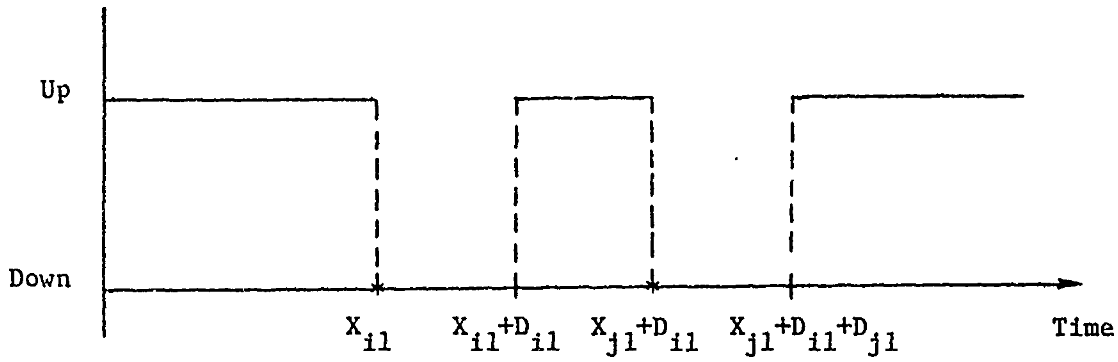



FIGURE 1: HISTORY OF A SERIES SYSTEM

In Figure 1, component $i$ fails at time $X_{i1}$ and the system is down $D_{i1}$ hours. The system again operates from time $X_{i1} + D_{i1}$ to time $X_{j1} + D_{i1}$. Component $j$ fails at time $X_{j1} + D_{i1}$ and is replaced by time $X_{j1} + D_{i1} + D_{j1}$.

Let $\xi(t) = i$ if the system is down at time $t$ due to the failure of component $i$ $(i = 1,2, ..., k)$. Let $\xi(t) = 0$ if the system is operating at time $t$. We are interested in the limiting probability, $\lim_{t\to\infty} P[\xi(t) = i]$, that the system is in state $i$. The limiting system availability, $\lim_{t\to\infty} P[\xi(t) = 0]$, is of special interest.

Since only failed components are replaced with new or like-new components, the age distribution of components in the system quickly becomes mathematically very complicated. The process $\{\xi(t) ; t \geq 0\}$ has in fact no regeneration points. It is remarkable, however, that many quantities of interest are, in the limit, mathematically very simple and depend only on component mean lives and component mean repair times. The limiting average system availability, as we shall prove in Section 2, is

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t P[\xi(u) = 0]du = 1\Big/\left[1 + \sum_{j=1}^{k} \nu_j/\mu_j\right] \overset{\text{def}}{=} \pi_0 \tag{0.1}$$

while

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t P[\xi(u) = i]du = \frac{\nu_i}{\mu_i}\pi_0 \overset{\text{def}}{=} \pi_i \ , \ i = 1,2, \ldots, k \ . \tag{0.2}$$

These formulae are true for arbitrary failure and repair distributions. If $\lim_{t\to\infty} P[\xi(t) = i]$ exists, then it is equal to $\pi_i$ . Although (0.1) is well known for exponential failure and exponential repair [cf. U.S. Naval Weapons Reliability Engineering Handbook (1968)], a rigorous proof seems to be missing for the general case. A heuristic proof for (0.1) was offered by Bazovsky, MacFarlane, and Wunderman (1962).

Let $\tilde{N}_i(t)$ be the number of replacements of component $i$ in time $t$ . We show that

$$\lim_{t\to\infty} \frac{E\tilde{N}_i(t)}{t} = \frac{\pi_0}{\mu_i} \ , \qquad i = 1,2, \ldots, k \ .$$

This result can be used in determining spare parts requirements, since $\pi_0 t/\mu_i$ will be, approximately, the number of components of type $i$ required in $[0,t]$ .

In Section 3 we prove the asymptotic normality of $t^{-\frac{1}{2}}\left[\tilde{N}_i(t) - tm_i^{-1}\right]$ where $m_i = \mu_i\pi_0^{-1}$ is, approximately, the mean time between failures of component $i$ . We also show that

$$t^{-\frac{1}{2}}[\tilde{N}(t) - tm^{-1}]$$

is asymptotically normal, where

$$\tilde{N}(t) = \sum_{i=1}^{k} \tilde{N}_i(t)$$

and $m^{-1} = \sum_{i=1}^{k} m_i^{-1}$ . A similar result ho[illegible] for $U(t)$ , system up time in $[0,t]$

$$t^{-\frac{1}{2}}[U(t) - \pi_0 t]$$

is asymptotically norma[illegible] [illegible] are used to calcula[illegible] repair and maintenance costs in Section 4.

## 1. PRELIMINARIES

Assume that all random variables associated with the series system discussed in the introduction are defined on a probability space $(\Omega, A, P)$. Recall that $\{X_{ir}(\cdot)\}_{r=1}^{\infty}$ and $\{D_{ir}(\cdot)\}_{r=1}^{\infty}$ are mutually independent renewal processes associated with component $i$ $(i = 1,2, \ldots, k)$. We suppress the argument of random variables, $X_{ir}(w)$, evaluated at $w \in \Omega$, except where useful in Section 2. Let $S_{in} = \sum_{r=1}^{n} X_{ir}$ and $N_i(t) = \sup\{n \mid S_{in} \le t\}$, where $S_{i0} \equiv 0$. Then $\{N_i(t) \; ; \; t \ge 0\}$ is the *renewal counting process* associated with $\{X_{ir}\}_{r=1}^{\infty}$. The following theorems are well known (cf. Feller, Vol. II, (1966) and Ross (1970)).

### 1.1 Theorem:

Let $\{N_i(t) \; ; \; t \ge 0\}$ be a renewal counting process corresponding to $\{X_{ir}\}_{r=1}^{\infty}$, where $EX_i = \mu_i$. Then

(1.1)
$$\lim_{t \to \infty} \frac{N_i(t)}{t} = \frac{1}{\mu_i} \qquad \text{a.s.},$$

and

(1.2)
$$\lim_{t \to \infty} \frac{EN_i(t)}{t} = \frac{1}{\mu_i} ,$$

where a.s. means almost surely with respect to $P$.

We will need the following generalization of the asymptotic normality of the renewal random variable, $N_i(t)$, in Section 3. Let $\{X_r\}_{r=1}^{\infty}$ be a sequence of nonnegative random variables, *not necessarily independent or identically distributed,* with an associated counting process $\{N(t) \; ; \; t \ge 0\}$ defined by

$$N(t) = \begin{cases} \sup\left\{n \mid \sum_{r=1}^{n} X_r \le t\right\} & X_1 \le t \\ 0 & X_1 > t \,. \end{cases}$$

The following theorem and its corollary can be proved by the argument in Feller, Volume I, (1968), p. 321.

1.2 Theorem:

$$(1.3) \qquad (\sigma^2 n)^{-\frac{1}{2}} \sum_{r=1}^{[n\tau]} (X_r - \mu) \to N(0,\tau)$$

as $n \to \infty$ if and only if

$$(1.4) \qquad (\sigma^2 \mu^{-3} n)^{-\frac{1}{2}} [N(n\tau) - n\tau/\mu] \to N(0,\tau) \,,$$

where [ ] means greatest integer contained within the brackets.

1.3 Corollary:

If $\{X_r\}_{r=1}^{\infty}$ is a renewal process with $EX_r = \mu$ and $\operatorname{Var} X_r = \sigma^2$, then both (1.3) and (1.4) hold.

Billingsley (1968), pp. 148-150, and Iglehart and Whitt (1969) generalize Theorem 1.2 to Wiener processes on $[0,1]$.

## 2. AVERAGE SYSTEM UP TIME: ALMOST SURE RESULTS

It will be useful to study the process $\{U(t) ; t \geq 0\}$ where $U(t)$ is the system functioning time (or up time) in $[0,t]$ . Similarly, let $D(t)$ be the down time in $[0,t]$ so that $U(t) + D(t) = t$ . Figure 2 is a very useful representation of a series system failure history in terms of system up time, $U(t)$ .

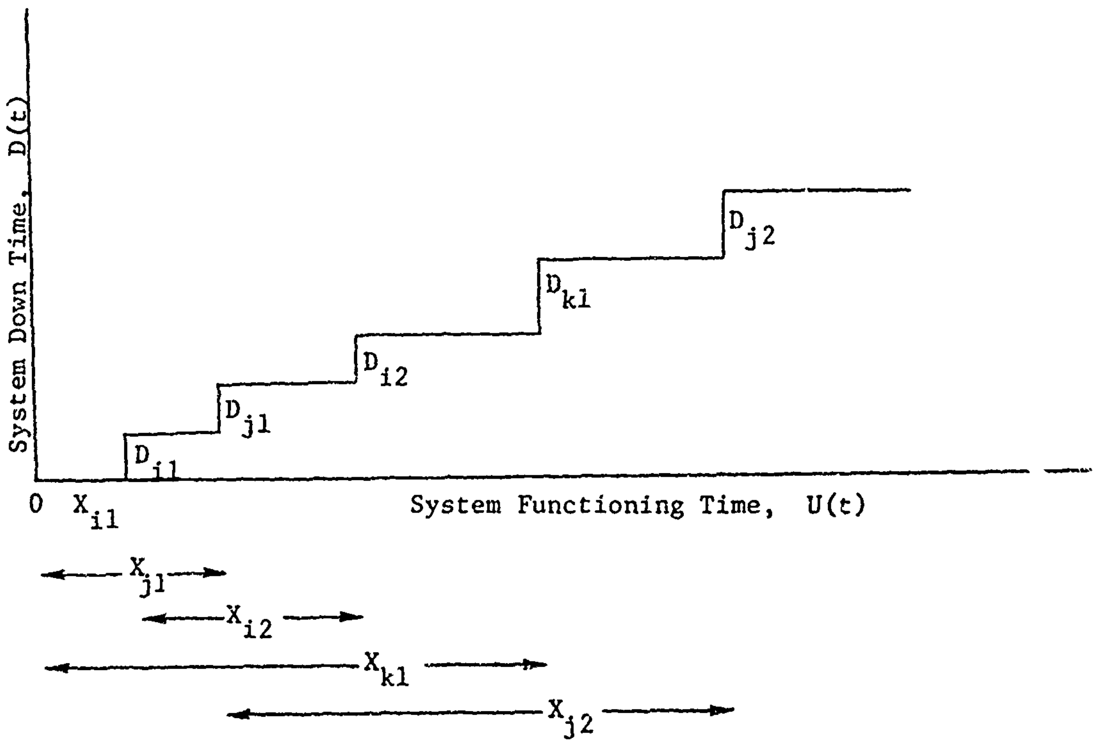



FIGURE 2

Let $N_i(t)$ be the number of failures of component $i$ in real time $t$ . Observe that

$$\tilde{N}_i(t) = N_i[U(t)] ,$$

where $\{N_i(t)\ ;\ t \geq 0\}$ is the renewal counting process associated with $\{X_{ir}\}_{r=1}^{\infty}$ . However, $N_i(\cdot)$ and $U(t)$ are *not* independent, since in particular $U(t) = t$ implies $N_i[U(t)] = 0$ . (We assume that all components are new at $t = 0$ for definiteness. However, the limiting results are true regardless of the initial conditions.)

2.1 Lemma:

If $0 < \mu_i < \infty$ and $0 \leq \nu_i < \infty$ $(i = 1,2, \ldots, k)$ , then

(2.1)
$$\lim_{t\to\infty} \frac{N_i[U(t,w),w]}{U(t,w)} = \frac{1}{\mu_i} \qquad \text{a.s.,}$$

where we have included the argument $w \in \Omega$ to emphasize that all random variables are defined on the same probability space $(\Omega, A, P)$ .

Proof:

Under the hypotheses $U(t,w) \to \infty$ almost surely as $t \to \infty$ . Since $N_i(t,w)$ and $U(t,w)$ are defined on the same probability space, (1.1) implies (2.1).||

2.2 Theorem:

If $0 < \mu_i < \infty$ and $0 \leq \nu_i < \infty$ $(i = 1,2, \ldots, k)$ , then

(2.2)
$$\lim_{t\to\infty} \frac{U(t)}{t} = \left[1 + \sum_{j=1}^{k} \nu_j/\mu_j\right]^{-1} \overset{\text{def}}{=} \pi_0 \qquad \text{a.s. .}$$

Proof:

Note that

$$\sum_{i=1}^{k} \sum_{r=1}^{\tilde{N}_i(t)-1} D_{ir} \leq D(t) \leq \sum_{i=1}^{k} \sum_{r=1}^{\tilde{N}_i(t)} D_{ir} \ .$$

The inequality results from the fact that the system may be down at time $t$.

Since $U(t) + D(t) = t$,

$$\frac{U(t)}{t} = \frac{1}{1 + \frac{D(t)}{U(t)}} \quad \frac{1}{1 + \sum_{i=1}^{k} \frac{1}{\tilde{N}_i(t)} \sum_{r=1}^{\tilde{N}_i(t)} D_{ir} \frac{N_i[U(t)]}{U(t)}} .$$

By the strong law, $\frac{1}{\tilde{N}_i(t)} \sum_{r=1}^{\tilde{N}_i(t)} D_{ir} \underset{a.s.}{\to} \nu_i$ as $t \to \infty$. By Lemma 2.1, $\frac{N_i[U(t)]}{U(t)} \underset{a.s.}{\to} \frac{1}{\mu_i}$ as $t \to \infty$. Hence,

$$\lim_{t\to\infty} \frac{U(t)}{t} \geq \frac{1}{1 + \sum_{i=1}^{k} \frac{\nu_i}{\mu_i}} = \pi_0 .$$

The reverse inequality is proved similarly.||

<u>2.3 Corollary</u>:

Under the same conditions,

$$\frac{EU(t)}{t} \to \pi_0 \quad \text{as} \quad t \to \infty .$$

<u>Proof</u>:

Since $\frac{U(t)}{t} \leq 1$ and $\frac{U(t)}{t} \underset{a.s.}{\to} \pi_0$, it follows by the Lebesgue dominated convergence theorem that

$$\frac{EU(t)}{t} \to \pi_0 \quad \text{as} \quad t \to \infty .||$$

2.4 Corollary:

Let $D_i(t)$ be the down time for component $i$ in $[0,t]$ . Then under the same hypotheses,

$$\lim_{t\to\infty} \frac{D_i(t)}{t} \overset{=}{\text{a.s.}} \frac{\nu_i}{\mu_i} \pi_0 \ . \tag{2.3}$$

Proof:

Note that

$$\sum_{r=1}^{\tilde{N}_i(t)-1} D_{ir} \le D_i(t) \le \sum_{r=1}^{\tilde{N}_i(t)} D_{ir} \ ,$$

so that

$$\frac{D_i(t)}{t} \le \frac{1}{\tilde{N}_i(t)} \left[\sum_{r=1}^{\tilde{N}_i(t)} D_{ir}\right] \frac{N_i[U(t)]}{U(t)} \frac{U(t)}{t} \ .$$

Hence,

$$\lim_{t\to\infty} \frac{D_i(t)}{t} \leqq \frac{\nu_i}{\mu_i} \pi_0$$

by the strong law, Lemma 2.1 and Theorem 2.2.

The reverse inequality is proved similarly.||

Of course, it also follows from Corollary 2.4 that

$$\lim_{t\to\infty} \frac{D(t)}{t} \overset{=}{\text{a.s.}} \pi_0 \sum_{i=1}^{k} \frac{\nu_i}{\mu_i} \ .$$

2.5 Corollary:

Under the same hypotheses,

$$\lim_{t\to\infty} \frac{\tilde{N}_i(t)}{t} \underset{\text{a.s.}}{=} \frac{\pi_0}{\mu_i} \qquad i = 1,2, \ldots, k \; . \tag{2.4}$$

Proof:

$$\frac{\tilde{N}_i(t)}{t} = \frac{N_i[U(t)]}{U(t)} \frac{U(t)}{t} \underset{\text{a.s.}}{\rightarrow} \frac{1}{\mu_i} \pi_0$$

by Lemma 2.1 and Theorem 2.2.||

2.6 Corollary:

Under the same hypotheses,

$$\lim_{t\to\infty} \frac{E\tilde{N}_i(t)}{t} = \frac{\pi_0}{\mu_i} \; . \tag{2.5}$$

Proof:

$$\frac{\tilde{N}_i(t)}{t} = \frac{N_i[U(t)]}{t} \leq \frac{N_i(t)}{t} \; .$$

By the elementary renewal theorem, (1.2)

$$\frac{EN_i(t)}{t} \rightarrow \frac{1}{\mu_i} \; .$$

Also, $EN_i(t) < \infty$ for all $t$. Hence, there exists $M$ such that $\sup_t \frac{EN_i(t)}{t} < M$. The conclusion follows from Corollary 2.5 and the Lebesgue dominated convergence theorem.||

Average Availability

We call $T^{-1}\int_0^T P[\xi(t) = 0]dt$ the *average availability* in $[0,T]$ . It is a well known property of stochastic processes that

$$(2.6)\qquad T^{-1}\int_0^T P[\xi(t) = 0]dt = \frac{EU(T)}{T} ,$$

an easy consequence of Fubini's Theorem. It follows from Corollary 2.3 that

$$(2.7)\qquad \lim_{T\to\infty} T^{-1}\int_0^T P[\xi(t) = 0]dt = \pi_0 .$$

If $\lim_{t\to\infty} P[\xi(t) = 0]$ exists, then it can easily be shown using (2.7) that

$$(2.8)\qquad \lim_{t\to\infty} P[\xi(t) = 0] = \pi_0 .$$

The limit in (2.8) will not always exist under the hypotheses of Theorem 2.2. Sufficient conditions for (2.8), for example, are $F_i$ nonlattice and $F_j$ exponential $(j \neq i)$ .

Similarly, if $P[\xi(t) = i]$ exists, then

$$(2.9)\qquad \lim_{t\to\infty} P[\xi(t) = i] = \pi_i .$$

System Mean Time Between Failures

Each time the system is repaired, the time until next failure will of course depend on the repair history of each component. However, the average of successive up times will converge to a limit, say $\mu$ . Likewise the average of successive down times will converge to a limit, say $\nu$ . To calculate these quantities, let $\tilde{N}(t) = \sum_{i=1}^{k} \tilde{N}_i(t)$ be the number of system failures in $[0,t]$ .

2.7 Theorem:

If $0 < \mu_i < \infty$ and $0 \le \nu_i < \infty$ $(i = 1,2, \ldots, k)$, then the limiting average of system up times will be a.s.

$$\mu = 1 \Big/ \sum_{i=1}^{k} \frac{1}{\mu_i} \tag{2.10}$$

while the limiting average of system down times will be a.s.

$$\nu = \mu \sum_{i=1}^{k} \nu_i/\mu_i \ . \tag{2.11}$$

Proof:

The average of system up times in $[0,t]$ will be approximately $U(t)/\tilde{N}(t)$. (The error will go to 0 a.s. as $t \to \infty$, as in previous proofs.) Since by Theorem 2.2, $\lim_{t\to\infty} U(t)/t \underset{a.s.}{=} \pi_0$, and by Corollary 2.5 $\lim_{t\to\infty} \frac{\tilde{N}_i(t)}{t} \underset{a.s.}{=} \pi_0/\mu_i$, it follows that

$$\mu \overset{def}{=} \lim_{t\to\infty} U(t)/\tilde{N}(t) \underset{a.s.}{=} \pi_0 \Big/ \left(\pi_0 \sum_{i=1}^{k} \frac{1}{\mu_i}\right) .$$

Hence,

$$\lim_{t\to\infty} U(t)/\tilde{N}(t) \underset{a.s.}{=} 1 \Big/ \sum_{i=1}^{k} \frac{1}{\mu_i} .$$

The average of system down times in $[0,t]$ will be approximately

$$\sum_{i=1}^{k} \sum_{r=1}^{\tilde{N}_i(t)} D_{ir}/\tilde{N}(t) .$$

By Corollary 2.5 and the strong law,

$$\sum_{i=1}^{k} \frac{\tilde{N}_i(t)}{\tilde{N}(t)} \frac{1}{\tilde{N}_i(t)} \sum_{r=1}^{\tilde{N}_i(t)} D_{ir}$$

$$\xrightarrow{\text{a.s.}} \mu \sum_{i=1}^{k} \frac{\nu_i}{\mu_i} \overset{\text{def}}{=} \nu \,. ||$$

<u>Remark</u>:

If failure distributions were exponential, then of course $1 \Big/ \sum_{i=1}^{k} \frac{1}{\mu_i}$ would be the expected duration of a system up time each time it is up, independently of system past history. It is interesting that in the limiting sense, the average duration of a system up time is

$$\mu = 1 \Big/ \sum_{i=1}^{k} \frac{1}{\mu_i}$$

for *arbitrary* failure distributions.

For a one unit system with mean life $\mu$ and mean repair time $\nu$, the limiting system fractional up time is

$$\mu/(\mu + \nu) \,. \tag{2.12}$$

For our series model, the limiting fractional system up time is, by Theorem 2.7,

$$\pi_0 = 1 \Bigg/ \left[1 + \sum_{i=1}^{k} \frac{\nu_i}{\mu_i}\right] = \mu \Bigg/ \left[\mu + \mu \sum_{i=1}^{k} \frac{\nu_i}{\mu_i}\right] = \mu/(\mu + \nu) \tag{2.13}$$

where now $\mu$ is defined by (2.10) and $\nu$ by (2.11). From Theorem 2.7, we see that (2.13) is the analogue of (2.12).

## 3. ASYMPTOTIC DISTRIBUTIONS

To obtain the asymptotic distribution of $\tilde{N}_i(t)$ , $\tilde{N}(t) = \sum_{i=1}^{k} \tilde{N}_i(t)$ , and $U(t)$ , we must also specify the variances $\sigma_i^2 = \mathrm{Var}\, X_i$ and $\tau_i^2 = \mathrm{Var}\, D_i$ $(i = 1,2, \ldots, k)$ . Let $m_i = \mu_i \pi_0^{-1}$ for $i = 1,2, \ldots, k$ . As we shall show, $m_i$ , is approximately the mean time between failures of component $i$ . Let $*$ denote normed random variables. Although normed random variables will have asymptotic mean $0$ , the asymptotic variance is not necessarily 1. In particular, let

$$\tilde{N}_i^*(t) = t^{-\frac{1}{2}}\left[\tilde{N}_i(t) - tm_i^{-1}\right] . \tag{3.1}$$

We first state our main results before presenting proofs.

### 3.1 Theorem:

If $0 < \mu_i < \infty$ , $0 \leq \nu_i < \infty$ , $0 < \sigma_i^2 < \infty$ , $0 \leq \tau_i^2 < \infty$ for $i = 1,2, \ldots, k$ , then

$$\left(\tilde{N}_1^*(t), \tilde{N}_2^*(t), \ldots, \tilde{N}_k^*(t)\right)$$

is asymptotically $(t \to \infty)$ multivariate normal with mean vector $0$ and variance-covariance matrix

$$\not{\Sigma} = (v_{ij}) ,$$

where

$$v_{ij} = m_i^{-1} m_j^{-1}\left[\pi_0 \sum_{s=1}^{k} \left[\nu_s^2 \sigma_s^2 \mu_s^{-3} + \tau_s^2 \mu_s^{-1}\right] - \nu_i \sigma_i^2 \mu_i^{-2} - \nu_j \sigma_j^2 \mu_j^{-2}\right] \qquad (i \neq j) \tag{3.2}$$

$$v_{ii} = v_i^2 = m_i^{-3} w_i^2 \tag{3.3}$$

and

$$w_i^2 = \sigma_i^2 c_i^2 + \mu_i \sum_{j=1}^{k} \tau_j^2 \mu_j^{-1} + \mu_i \sum_{j \neq i} \sigma_j^2 \nu_j^2 \mu_j^{-3} ,$$

and

$$c_i = 1 + \sum_{j \neq i} \nu_j \mu_j^{-1} , \quad \pi_0 = \left[ 1 + \sum_{j=1}^{k} \nu_j \mu_j^{-1} \right]^{-1} . \tag{3.4}$$

3.2 Corollary:

Under the conditions of Theorem 3.1,

$$t^{-\frac{1}{2}} v_i^{-1} \left[ \tilde{N}_i(t) - t m_i^{-1} \right]$$

is asymptotically $(t \to \infty)$ $N(0,1)$ .

3.3 Theorem:

Let $\tilde{N}(t) = \sum_{i=1}^{k} \tilde{N}_i(t)$ and $m^{-1} = \sum_{i=1}^{k} m_i^{-1}$ . Under the conditions of Theorem 3.1,

$$t^{-\frac{1}{2}} v^{-1} [\tilde{N}(t) - t m^{-1}]$$

is asymptotically $(t \to \infty)$ $N(0,1)$ , where

$$v^2 = \pi_0 m^{-2} \sum_{i=1}^{k} \left[ (m - \nu_i)^2 \sigma_i^2 \mu_i^{-3} + \tau_i^2 \mu_i^{-1} \right] . \tag{3.5}$$

3.4 Theorem:

Let $U(t)$ be the system up time in $[0,t]$ . Under the conditions of Theorem 3.1,

$$t^{-\frac{1}{2}}u^{-1}[U(t) - \pi_0 t]$$

is asymptotically $(t \to \infty)$ , $N(0,1)$ where

$$u^2 = \pi_0^3 \left[ \sum_{i=1}^{k} \left( \nu_i^2 \mu_i^{-3} \sigma_i^2 + \tau_i^2 \mu_i^{-1} \right) \right] . \tag{3.6}$$

3.5 Corollary:

Let $D(t)$ be the system down time in $[0,t]$ so that $U(t) + D(t) = t$ . Under the conditions of Theorem 3.1

$$t^{-\frac{1}{2}}u^{-1}[D(t) - (1 - \pi_0)t]$$

and

$$t^{-\frac{1}{2}}\left[\tau_i^2 m_i^{-1} + \nu_i^2 w_i^2 m_i^{-3}\right]^{-\frac{1}{2}}[D_i(t) - \pi_i t]$$

are asymptotically $(t \to \infty)$ $N(0,1)$ where $D_i(t)$ is the down time of component $i$ in $[0,t]$ .

Applications of these results to the problem of determining maintenance costs are discussed in Section 4.

Proofs of Theorems:

For mathematical convenience, we will work with the random variable which is the number of *completed repairs* of component $i$ in $[0,t]$ . We do this because we want to assume that all components are new at $t = 0$ . A natural cycle, then, ends with a completed repair. Asymptotically, the number of failures properly normed will have the same distribution as the number of completed repairs similarly normed. For this reason we use $\tilde{N}_i(t)$ to mean the number of completed repairs in $[0,t]$ for the remainder of this section.

Let $S_{in} = \sum_{r=1}^{n} X_{ir}$ and $S_{i0} = 0$ . The time to the first completed repair of component $i$ will then be

$$Y_{i1} = X_{i1} + D_{i1} + \sum_{j \neq i} \sum_{s=1}^{N_j(X_{i1})} D_{js}$$

where $\{N_j(x); x \geq 0\}$ is the renewal counting process associated with $\{X_{jr}\}_{r=1}^{\infty}$ as in Section 1. Similarly, the time between the $r-1$ and rth completed repair will be

$$Y_{ir} = X_{ir} + D_{ir} + \sum_{j \neq i} \sum_{s=N_j(S_{i,r-1})+1}^{N_j(S_{i,r})} D_{js} . \tag{3.7}$$

Let

$$Z_{in} = \sum_{r=1}^{n} Y_{ir} = S_{in} + \sum_{r=1}^{n} D_{ir} + \sum_{j \neq i} \sum_{r=1}^{N_j(S_{in})} D_{jr} . \tag{3.8}$$

3.6 Lemma:

If $0 < \mu_i < \infty$ and $0 \leq \nu_i < \infty$ $(i = 1,2, \ldots, k)$ , then

$$\lim_{n \to \infty} n^{-1} Z_{in} \underset{a.s.}{=} \mu_i \pi_0^{-1} \overset{def}{=} m_i .$$

Proof:

By the strong law $n^{-1} S_{in} \underset{a.s.}{\to} \mu_i$ and $n^{-1} \sum_{r=1}^{n} D_{ir} \underset{a.s.}{\to} \nu_i$ . Since $\mu_i > 0$ , $S_{in} \underset{a.s.}{\to} \infty$ as $n \to \infty$ , and since $\mu_j > 0$ , $N_j(t) \underset{a.s.}{\to} \infty$ as $t \to \infty$ . Hence

$$\left[\frac{1}{N_j(S_{in})}\sum_{s=1}^{N_j(S_{in})} D_{js}\right]\left[\frac{N_j(S_{in})}{S_{in}}\right]\left[\frac{S_{in}}{n}\right] \overset{\rightarrow}{a.s.} \nu_j\mu_j^{-1}\mu_i$$

again by the strong law and (2.3). The conclusion of the lemma follows.||

The processes $\{Z_{in};n \geq 1\}$ and $\{\tilde{N}_i(t);t \geq 0\}$ are related by

$$\tilde{N}_i(t) = \begin{cases} \max\ \{n \mid Z_{in} \leq t\}\ , & Z_{i1} \leq t \\ 0 & Z_{i1} > t\ . \end{cases} \tag{3.9}$$

Since the partial sum process $\{Z_{in};n \geq 1\}$ and the counting process $\{N_i(t);t \geq 0\}$ are essentially inverses of each other, it will be sufficient to determine the asymptotic normality of the partial sum process.

From (3.9) we observe that for fixed $\tau > 0$

$$\begin{aligned} m_i^{-1}n^{-\frac{1}{2}}\sum_{r=1}^{\tilde{N}_i(n\tau)} [Y_{ir} - m_i] &\leq n^{-\frac{1}{2}}\left[\frac{n\tau}{m_i} - \tilde{N}_i(n\tau)\right] \\ &\leq m_i^{-1}n^{-\frac{1}{2}}\sum_{r=1}^{N_i(n\tau)} [Y_{ir} - m_i] + m_i^{-1}n^{-\frac{1}{2}}Y_{i,\tilde{N}_i(n\tau)+1}\ , \end{aligned} \tag{3.10}$$

so that asymptotically

$$n^{-\frac{1}{2}}\left[n\tau m_i^{-1} - \tilde{N}_i(n\tau)\right] \sim m_i^{-1}n^{-\frac{1}{2}}\sum_{r=1}^{\tilde{N}_i(n\tau)} [Y_{ir} - m_i]\ , \tag{3.11}$$

where $\sim$ means asymptotic equivalent in distribution. On the other hand, it is well known (Renyi (1957)) that

$$n^{-\frac{1}{2}}\sum_{r=1}^{N_i(n\tau)} [Y_{ir} - m_i] \sim n^{-\frac{1}{2}}\sum_{r=1}^{\left[\frac{n\tau}{m_i}\right]} [Y_{ir} - m_i] \tag{3.12}$$

since $\dfrac{N_i(n\tau)}{n\tau} \underset{\text{a.s.}}{\rightarrow} 1/m_i$ by (2.4).

It will be useful to expand $Z^*_{in} \overset{\text{def}}{=} n^{-\frac{1}{2}} \sum_{r=1}^{n} [y_{ir} - m_i]$ as follows:

$$Z^*_{in} = n^{-\frac{1}{2}}(S_{in} - n\mu_i) + n^{-\frac{1}{2}} \sum_{r=1}^{n} (D_{ir} - \nu_i)$$

$$+ n^{-\frac{1}{2}} \sum_{j\neq i} \sum_{r=1}^{N_j(S_{in})} (D_{jr} - \nu_j) + n^{-\frac{1}{2}} \sum_{j\neq i} \nu_j\left[N_j(S_{in}) - \frac{S_{in}}{\mu_j}\right]$$

$$+ n^{-\frac{1}{2}} \sum_{j\neq i} \frac{\nu_j}{\mu_j} (S_{in} - n\mu_i) \; .$$

Let $S^*_{in} = n^{-\frac{1}{2}}(S_{in} - n\ _i)$ and $c_i = 1 + \sum_{j\neq i} \nu_j\mu_j^{-1}$ . Then we can rewrite $Z^*_{in}$ as

$$Z^*_{in} = c_i S^*_{in} + n^{-\frac{1}{2}} \sum_{r=1}^{n} (D_{ir} - \nu_i)$$

$$+ n^{-\frac{1}{2}} \sum_{j\neq i} \sum_{r=1}^{N_j(S_{in})} (D_{jr} - \nu_j) + n^{-\frac{1}{2}} \sum_{j\neq i} \nu_j\left[N_j(S_{in}) - \frac{S_{in}}{\mu_j}\right] .$$

Also

$$Z^*_{in} \sim c_i S^*_{in} + n^{-\frac{1}{2}} \sum_{j=1}^{k} \sum_{r=1}^{[n\mu_i\mu_j^{-1}]} (D_{jr} - \nu_j)$$

(3.13)

$$+ \sum_{j\neq i} n^{-\frac{1}{2}}\nu_j\left[N_j(n\mu_i) - n\mu_i\mu_j^{-1}\right] .$$

Note that the summands in (3.13) are now independent.

3.7 Lemma:

$$\lim_{n\to\infty} \operatorname{Var} Z^*_{in} = c_i^2\sigma_i^2 + \mu_i \sum_{j=1}^{k} \tau_j^2\mu_j^{-1} + \mu_i \sum_{j\neq i} \sigma_j^2\nu_j^2\mu_j^{-3} \tag{3.14}$$

$$\stackrel{\text{def}}{=} w_i^2 ,$$

where $c_i = 1 + \sum_{j\neq i} \nu_j/\mu_j$ .

Proof:

Use representation (3.13) and the well-known result

$$\operatorname{Var} n^{-\frac{1}{2}}[N_j(n\tau) - n\tau/\mu_j] \to \sigma_j^2\mu_j^{-3}\tau \ .||$$

We use (3.11) and (3.12) to write

$$n^{-\frac{1}{2}}\nu_j\left[N_j(n\mu_i) - n\mu_i\mu_j^{-1}\right] - \frac{-\nu_j}{\mu_j} S^*_{j\left[n\mu_i\mu_j^{-1}\right]} \ . \tag{3.15}$$

Proof of Theorem 3.1:

From Theorem 1.2 and representation (3.13) it is obvious that the marginal random variables $\tilde{N}_i^*(t)$ are asymptotically $N\left(0,w_i^2m_i^{-3}\right)$ where $w_i^2$ is given by (3.14). Using (3.15) we see that

$$Z^*_{in} - c_iS^*_{in} - \sum_{j\neq i} \nu_j\mu_j^{-1}S^*_{j\left[n\mu_i\mu_j^{-1}\right]} + n^{-\frac{1}{2}} \sum_{j=1}^{k} \sum_{r=1}^{\left[n\mu_i\mu_j^{-1}\right]} (D_{jr} - \nu_j)$$

or

$$Z^*_{in} - \pi_0^{-1}S^*_{in} - \sum_{j=1}^{k} \nu_j\mu_j^{-1}S^*_{j\left[n\mu_i\mu_j^{-1}\right]} + n^{-\frac{1}{2}} \sum_{j=1}^{k} \sum_{r=1}^{\left[n\mu_i\mu_j^{-1}\right]} (D_{jr} - \nu_j) \ . \tag{3.16}$$

From representation (3.16) it is easy to see that aribtrary linear combinations of the $Z^*_{in}$'s are asymptotically normal. It follows that $\tilde{N}^*_1(t), \tilde{N}^*_2(t), \ldots, \tilde{N}^*_k(t)$ is asymptotically multivariate normal.

It remains to compute the variance-covariance matrix. From Theorem 1.2 and Lemma 3.7, it follows that

$$v_{ii} = v_i^2 = m_i^{-3} w_i^2$$

as given by (3.3).

To compute $\mathrm{Cov}\left[\tilde{N}^*_i(t), \tilde{N}^*_j(t)\right]$ for $i \neq j$, recall that by (3.15)

$$\tilde{N}^*_i(n\tau) \sim -m_i^{-1} Z^*_{i\left[\frac{n\tau}{m_i}\right]} .$$

Hence,

$$v_{ij} \overset{\text{def}}{=} \mathrm{Cov}\left[\tilde{N}^*_i(n\tau), \tilde{N}^*_j(n\tau)\right] = m_i^{-1} m_j^{-1} \mathrm{Cov}\left[Z^*_{i\left[\frac{n\tau}{m_i}\right]}, Z^*_{j\left[\frac{n\tau}{m_j}\right]}\right]$$

for $0 \leq \tau \leq 1$ . Using (3.16) we see that

$$\text{(3.17)} \quad Z^*_{i\left[\frac{n\tau}{m_i}\right]} \sim \pi_0^{-1} S^*_{i\left[\frac{n\tau\pi_0}{\mu_i}\right]} - \sum_{s=1}^{k} \nu_s \mu_s^{-1} S^*_{s\left[\frac{n\tau\pi_0}{\mu_s}\right]} + n^{-\frac{1}{2}} \sum_{s=1}^{k} \sum_{r=1}^{\left[n\tau\pi_0\mu_s^{-1}\right]} (D_{sr} - \nu_s) .$$

It follows that for $i \neq j$ ,

$$\mathrm{Cov}\left[Z^*_{i\left[\frac{n\tau}{m_i}\right]}, Z^*_{j\left[\frac{n\tau}{m_j}\right]}\right]$$

$$= -\pi_0^{-1}\frac{\nu_i}{\mu_i}\,\mathrm{Var}\left(S^*_{i\left[\frac{n\tau\pi_0}{\mu_i}\right]}\right) - \pi_0^{-1}\frac{\nu_j}{\mu_j}\,\mathrm{Var}\left(S^*_{j\left[\frac{n\tau\pi_0}{\mu_j}\right]}\right)$$

$$+ \sum_{s=1}^{k} \nu_s^2\mu_s^{-2}\,\mathrm{Var}\left(S^*_{s\left[\frac{n\tau\pi_0}{\mu_s}\right]}\right) + \sum_{s=1}^{k} \pi_0\mu_s^{-1}\tau_s^2\tau \ .$$

Hence, letting $\tau = 1$ ,

$$v_{ij} = m_i m_j\left[-\nu_i\mu_i^{-2}\sigma_i^2 - \nu_j\mu_j^{-2}\sigma_j^2 + \pi_0\sum_{s=1}^{k}\left[\nu_s^2\mu_s^{-3}\sigma_s^2 + \tau_s^2\mu_s^{-1}\right]\right] \ . ||$$

Proof of Theorem 3.3:

From the previous proof we know that

$$\tilde{N}_n^*(n\tau) \overset{\text{def}}{=} n^{-\frac{1}{2}}[N(n\tau) - n\tau m^{-1}] \sim -\sum_{i=1}^{k} m_i^{-1} Z^*_{i\left[\frac{n\tau}{m_i}\right]} \qquad 0 \le \tau \le 1 \ .$$

From (3.17) we see that

$$\tilde{N}_n^*(n\tau) \sim -\pi_0^{-1}\sum_{i=1}^{k} m_i^{-1} S^*_{i\left[\frac{n\tau\pi_0}{\mu_i}\right]} + m^{-1}\sum_{j=1}^{k}\nu_j\mu_j^{-1} S^*_{j\left[\frac{n\tau\pi_0}{\mu_j}\right]}$$

(3.18)

$$- m^{-1}n^{-\frac{1}{2}}\sum_{j=1}^{k}\sum_{r=1}^{\left[n\tau\pi_0\mu_j^{-1}\right]}(D_{jr} - \nu_j) \ .$$

(3.18) is clearly asymptotically normal with variance

$$v^2 = \pi_0 m^{-2} \sum_{j=1}^{k} \left[(m - \nu_j)^2 \mu_j^{-3} \sigma_j^2 + \tau_j^2 \mu_j^{-1}\right]$$

at $\tau = 1$ .

Proof of Theorem 3.4:

By definition of the up time, $U(t)$ , in $[0,t]$ ,

$$\sum_{r=1}^{\tilde{N}_i(t)} X_{ir} \le U(t) \le \sum_{r=1}^{\tilde{N}_i(t)} X_{ir} + X_{i,\tilde{N}_i(t)+1} .$$

Since $\sup_{\tau \le 1} \left(X_{i,\tilde{N}_i(\tau n)+1}\right)/\sqrt{n}$ converges to 0 in probability, it follows that

$$U_n^*(n\tau) \overset{\text{def}}{=} n^{-\frac{1}{2}}[U(n\tau) - n\tau\pi_0] \sim n^{-\frac{1}{2}}\left[\sum_{r=1}^{\tilde{N}_i(n\tau)} (X_{ir} - \mu_i) + \mu_i\left[\tilde{N}_i(n\tau) - \frac{n\tau}{m_i}\right]\right]$$

for $0 \le \tau \le 1$ .

As in previous proofs

$$n^{-\frac{1}{2}}\mu_i\left[\tilde{N}_i(n\tau) - \frac{n\tau}{m_i}\right] \sim -\pi_0 Z^*_{i\left[\frac{n\tau}{m_i}\right]} .$$

Hence

$$U_n^*(n\tau) \sim S^*_{i\left[\frac{n\tau}{m_i}\right]} - \pi_0 Z^*_{i\left[\frac{n\tau}{m_i}\right]} .$$

Using (3.17) we see that

$$U_n^*(n\tau) \sim \pi_0 \sum_{j=1}^{k} \nu_j \mu_j^{-1} S^*_{j\left[n\tau\pi_0\mu_j^{-1}\right]} - \pi_0 n^{-\frac{1}{2}} \sum_{j=1}^{k} \sum_{r=1}^{\left[n\tau\pi_0\mu_j^{-1}\right]} (D_{jr} - \nu_j) .$$

Clearly $U_n^*(n\tau)$ is asymptotically normal with variance

$$u^2 = \pi_0^3 \sum_{j=1}^{k} \left[\nu_j^2 \mu_j^{-3} \sigma_j^2 + \tau_j^2 \mu_j^{-1}\right]$$

when $\tau = 1$ . ||

## 4. COST OF REPAIR

First suppose that component i costs $d_i$ *dollars to repair each time it fails* regardless of the time to complete repair. Then the cost accrued during $[0,t]$ is

$$C_1(t) = \sum_{i=1}^{k} d_i \tilde{N}_i(t)$$

and the cost per unit of time is, in the limit,

$$\lim_{t\to\infty} \frac{C_1(t)}{t} = \pi_0 \sum_{i=1}^{k} d_i \mu_i^{-1} \ . \tag{4.1}$$

Applying the techniques used in the proof of Theorem 3.3, we can show that

$$t^{-\frac{1}{2}}\left[C_1(t) - \pi_0 \sum_{i=1}^{k} d_i \mu_i^{-1}\right] \to N\left(0,\delta_1^2\right) \ , \tag{4.2}$$

where

$$\delta_1^2 = \pi_0 \sum_{i=1}^{k} \left[d_i + \left(\sum_{j=1}^{k} d_j m_j^{-1}\right)\nu_i\right]^2 \mu_i^{-3}\sigma_i^2$$
$$+ \pi_0\left(\sum_{j=1}^{k} d_j m_j^{-1}\right)^2 \sum_{j=1}^{k} \tau_j^2 \mu_j^{-1} \ .$$

Alternatively, suppose that it costs $d_i$ *dollars per hour of down time* for component i . Now the total cost accrued during $[0,t]$ becomes

$$C_2(t) = \sum_{i=1}^{k} d_i D_i(t)$$

where $D_i(t)$ is the down time for component i in $[0,t]$ . Then

$$\lim_{t\to\infty} \frac{C_2(t)}{t} = \pi_0 \sum_{i=1}^{k} d_i \nu_i \mu_i^{-1} \tag{4.3}$$

by (2.7).

4.1 Theorem:

Under the conditions of Theorem 3.1,

$$t^{-\frac{1}{2}}\left[C_2(t) - t\pi_0 \sum_{i=1}^{k} d_i \nu_i \mu_i^{-1}\right] \to N\left(0, \delta_2^2\right) \tag{4.4}$$

where

$$\delta_2^2 = \sum_{i=1}^{k} \left[\tau_i^2 m_i^{-1} + \nu_i^2 m_i^{-3} w_i^2\right] d_i^2 ,$$

where $w_i^2$ is given in Lemma 3.7.

Proof:

$$n^{-\frac{1}{2}}\left[C_2(n\tau) - n\tau\pi_0 \sum_{i=1}^{k} d_i \nu_i \mu_i^{-1}\right]$$

$$\sim n^{-\frac{1}{2}} \sum_{i=1}^{k} \sum_{r=1}^{\tilde{N}_i(n\tau)} d_i (D_{ir} - \nu_i) + n^{-\frac{1}{2}} \sum_{i=1}^{k} \nu_i d_i \left[\tilde{N}_i(n\tau) - n\tau m_i^{-1}\right]$$

$$\sim n^{-\frac{1}{2}} \sum_{i=1}^{k} \sum_{r=1}^{n\tau m_i^{-1}} d_i (D_{ir} - \nu_i) - \sum_{i=1}^{k} \nu_i d_i m_i^{-1} Z^*_{i\left[n\tau m_i^{-1}\right]} .$$

Obviously, the normed cost function is asymptotically normal. Its variance is easily seen to be (when $\tau = 1$)

$$\delta_2^2 = \sum_{i=1}^{k} \left[m_i^{-1} \tau_i^2 + \nu_i^2 m_i^{-3} w_i^2\right] d_i^2 \,.||$$

## REFERENCES


[1] Bazovsky, I., N. R. Macfarlane and R. Wunderman, "Study of Maintenance Cost Optimization and Reliability of Shipboard Machinery," United Control Report for ONR Contract No. Nonr-37400, Seattle, Washington, (1962).

[2] Billingsley, P., CONVERGENCE OF PROBABILITY MEASURES, John Wiley & Sons, (1968).

[3] Botez, M. C., "Sur un Modele Reparation de L'Equipment," Cahiers du Centre d'Etudes de Recherche Operationnelle, c/o Institute de Statistique de l'Universite Libre de Bruxelles, Vol. 11, No. 2, (1969).

[4] Colloquium on Reliability Theory, Tihany, Hungary, (September 16-19, 1969).

[5] Feller, W., AN INTRODUCTION TO PROBABILITY THEORY AND ITS APPLICATIONS, Vol. II, John Wiley & Sons, (1966).

[6] Feller, W., AN INTRODUCTION TO PROBABILITY THEORY AND ITS APPLICATIONS, Vol. I, John Wiley and Sons, (3rd Edition), (1968).

[7] Gaver, D. P., "Time to Failure and Availability of Paralleled Systems with Repair," IEEE Transactions on Reliability, Vol. R-12, pp. 30-38, (June 1963).

[8] HANDBOOK OF RELIABILITY ENGINEERING, NAVAIR, 00-65-502, published by Ordinance Systems Command, (March 1968).

[9] Iglehart, D. and W. Whitt, "The Equivalence of Functional Central Limit Theorems for Counting Processes and Associated Partial Sums," Technical Report No. 121, Stanford University, Department of Operations Research, (1969).

[10] Obretenov, A., B. Dimitrov and M. Uzunov, "Study on the Reliability of a System by Means of Stochastic Processes," (in Russian), Bulgarian Academy of Sciences, Bulletin of the Institute of Mathematics, Vol. XI, pp. 159-180, (1969).

[11] Renyi, A., "On the Asymptotic Distribution of the Sum of a Random Number of Independent Random Variables," Acta. Math. Acad. Sci. Hung., Vol. 8, pp. 193-199, (1957).

[12] Ross, S., APPLIED PROBABILITY MODELS WITH OPTIMIZATION APPLICATIONS, Holden-Day, San Francisco, California, (1970).